JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **159**, 44–50 (1991)

A Different Consideraton about the Globally Asymptotically Stable Solution of the Periodic n-Competing Species Problem*

ANTONIO TINEO AND CARLOS ALVAREZ

Departamento de Matematicas, Facultad de Ciencias, Universidad de Los Andes, Merida, Venezuela

Submitted by John L. Casti

Received August 10, 1988

We consider the Volterra–Lotka equations for n-competing species ($n \geqslant 2$) in which the right-hand sides are periodic in time. We show that conditions given by K. Gopalsamy (*J. Austral. Math. Soc. Ser. B* **27**, 1985, 66–72), which imply the existence of a periodic solution with positive components, also imply the uniqueness and asymptotic stability of the solution. © 1991 Academic Press, Inc.

INTRODUCTION

Let us consider the Volterra–Lotka equations

$$u_i' = u_i \left[b_i(t) - \sum_{j=1}^{n} a_{ij}(t)\, u_j \right]; \qquad 1 \leqslant i \leqslant n, \tag{0.1}$$

when $n \geqslant 2$ and a_{ij}, b_i: $\mathbb{R} \to R$ are continuous, positive, and T-periodic for some common period $T > 0$.

In [4] Gopalsamy has shown that the conditions

$$\min(b_i) > \sum_{j \in J_i} \frac{\max(a_{ij})}{\min(a_{jj})} \max(b_j); \qquad 1 \leqslant i \leqslant n, \tag{0.2}$$

where $J_i = \{1, ..., i-1, i+1, ..., n\}$, imply the existence of a T-periodic solution u^0 of (0.1) all of whose components are positive. In [4] it was also shown that if the inequalities (0.2) hold and if, in addition,

$$\min(a_{ii}) > \sum_{j \in J_i} \max(a_{ji}); \qquad 1 \leqslant i \leqslant n \tag{0.3}$$

* This article was sponsored by the C.D.C.H.T. Universidad de Los Andes, Venezuela.

0022-247X/91 $3.00

then, $u(t)-u^0(t)\to 0$ as $t\to +\infty$, for any solution u of (0.1) whose component has positive initial values.

We shall prove that the second assertion of the Gopalsamy Theorem [4] is a consequence of conditions (0.2). This result was shown by Alvarez and Lazer [2] in the case $n=2$.

For each $i=1, ..., n$ let us denote by U_i^0 the unique T-periodic and positive solution of the logistic equation

$$U' = U[b_i(t) - a_{ii}(t)\,U]. \tag{0.4$_i$}$$

Our main result is the following.

0.1. THEOREM. *Suppose that*

$$b_i(t) > \sum_{j\in J_i} a_{ij}(t)\,U_j^0(t); \qquad 1\leqslant i\leqslant n; \quad t\in\mathbb{R} \tag{0.5}$$

then the system (0.1) *has a T-periodic solution u^0 of whose components are positive. Moreover, if there are positive constants $\alpha_1, ..., \alpha_n$ such that*

$$\alpha_i a_{ii}(t) > \sum_{j\in J_i} \alpha_j a_{ji}(t); \qquad 1\leqslant i\leqslant n; \quad t\in\mathbb{R} \tag{0.6}$$

then $u(t)-u^0(t)\to 0$ as $t\to\infty$, for any positive solution u of (0.1).

Remark. It is easy to prove that $\max(U_i^0)\leqslant \max(b_i/a_{ii})$ and hence (0.2) implies (0.5). We shall prove that the condition (0.2) also implies condition (0.6). So the Gopalsamy Theorem [4] is true only under the condition (0.2).

1. Some Corollaries to the Main Result

In this section we get some corollaries to Theorem 0.1. In particular we shall prove that the Gopalsamy Theorem [4] remains true under conditions (0.2). We will get also a generalization of the main result of Alvarez and Lazer [2].

Given $x=(x_1, ..., x_n)\in\mathbb{R}^n$ we put $x>0$ if $x_i>0$; $i=1, ..., n$.

The points of $\mathbb{R}^n$ are considered as columns vectors; $x=\mathrm{col}(x_1, ..., x_n)$.

1.1. THEOREM. *Suppose that the conditions* (0.2) *are satisfied, then the system* (0.1) *has T-periodic solution u^0 all of whose components are positive and $u(t)-u^0(t)\to 0$, as $t\to 0$, for any positive solution u of* (0.1).

Proof. We know that the conditions (0.2) imply conditions (0.5); so it is suffices to prove that conditions (0.2) also imply conditions (0.6). To prove this let $B=(b_{ij})$ be the real $n\times n$-matrix defined by $b_{ii}=0$ and $b_{ij}=\max(a_{ij})/\max(a_{jj})$ if $i\neq j$. Thus $B\beta<\beta$ where $\beta=\mathrm{col}(\min(b_1), ..., \min(b_n))$. Let us fix $\varepsilon>n$ such that $B_\varepsilon\beta<\beta$ where $B_\varepsilon=B+\varepsilon$ (identity); from Perron's theorem we get a real positive eigenvalue λ of B_ε such that $\lambda<1$ and $|\mu|\leqslant\lambda$ for all eigenvalues μ of B_ε (see [3, p. 277] for details). Once again, from Perron's theorem we have $B_\varepsilon^*\alpha=\lambda\alpha$ for some $\alpha>0$, where B_ε^* is the adjoint matrix of B_ε. Therefore $B^*\alpha=(\lambda-\varepsilon)\alpha<\alpha$ which implies (0.6), and the proof is complete.

1.2. *Remark.* Let $B=(b_{ij})$ be the $n\times n$-matrix given by $b_{ij}=0$ and $b_{ij}=\max(a_{ji}/a_{ii})$ if $i\neq j$. If B has no eigenvalues in $[1,\infty)$ then condition (0.6) is satisfied.

Proof. From Perron's theorem we have an eigenvalue λ of B such that $B\alpha=\lambda\alpha$ for some $\alpha>0$. Since $\lambda<1$ we get $B\alpha<\alpha$, which implies (0.6); and so the proof is finished.

1.3. COROLLARY. *Let B be as in Remark* 1.2 *and assume that* $2\leqslant n\leqslant 3$ *and* $\det(I-B)>0$ ($I=$ *identity matrix*). *If conditions* (0.5) *are satisfied then the assertions of Theorem* 1.1 *are true.*

Proof. It is easy to verify that the conditions $2\leqslant n\leqslant 3$ and $\det(I-B)>0$ imply that B has no eigenvalues in $[1,\infty)$. The proof follows now from Remark 1.2 and Theorem 0.1.

1.4. COROLLARY. *Suppose that $n=2$ and*

$$\max(a_{12}/b_1)\max(b_2/a_{22})<1 \tag{1.1}$$

$$\max(a_{21}/b_2)\max(b_1/a_{11})<1 \tag{1.2}$$

$$\max(a_{21}/a_{11})\max(a_{12}/a_{22})<1. \tag{1.3}$$

Then the assertions of Theorem 1.1 *are true.*

Proof. It is easy to prove that the conditions (1.1)–(1.2) imply conditions (0.5). Moreover, (1.3) is equivalent to $\det(I-B)>0$ where B is as in Remark 1.2. Thus, the proof follows Corollary 1.3.

Remarks. (a) Corollary 1.4 was proved by Alvarez and Lazer [2] under the conditions

$$\max(b_1)\max(a_{21})<\min(a_{11})\min(b_2) \tag{1.4}$$

$$\max(b_2)\max(a_{12})<\min(a_{22})\min(b_1). \tag{1.5}$$

It is clear that (1.4) and (1.5) imply (1.2) and (1.1), respectively. On the other hand, (1.4) and (1.5) imply that $\min(a_{11})\min(a_{22})>\max(a_{12})\max(a_{21})$ which implies (1.3).

Finally it is easy to give many examples where conditions (1.1)–(1.3) are fulfilled but the conditions (1.4)–(1.5) are not satisfied. So Corollary 1.4 is a generalization of the main result of [2].

(b) Following the ideas of Ahmad [1], Corollary 1.4 can be proved in the almost-periodic case.

2. EXISTENCE

In this section we shall prove the existence claim of Theorem 0.1. To prove this we need two intermediate results.

2.1. PROPOSITION. *Let $u=(u_1, ..., u_n)$ be a solution of* (0.1) *with $u_i>0$; $1\leqslant i\leqslant n$; and let us fix t_0 in the domain of u. For each $i=1, ..., n$ let U_i be a solution of* (0.4) *such that $U_i(t_0)\geqslant u_i(t_0)$, then $U_i(t)>u_i(t)$ if $t>t_0$; $1\leqslant i\leqslant n$.*

Proof. Let us fix $i=1, ..., n$. If $U_i(t)=u_i(t_0)$ then $U_i'(t_0)>u_i'(t_0)$. Therefore, if $U_i(t_0)\geqslant u_i(t_0)$, then there exists $t_1>t_0$ such that $U_i>u_i$ in (t_0, t_1) and the proof follows from classical arguments.

Remark. If $u=(u_1, ..., u_n)$ is a positive T-periodic solution of (0.1) then $u_i<U_i^0$; $1\leqslant i\leqslant n$.

Proof. Suppose that $u_i(t_0)=U_1^0(t_0)$ for some $i=1, ..., n$ and some $t_0\in\mathbb{R}$. From Proposition 2.1 we have that $U_i>u_i$ in (t_0, ∞) which contradicts the fact that $U_i^0-u_i$ is T-periodic. Assume now that $u_i>U_i$ for some $i=1, ..., n$. Then $(u_i/U_i^0)'<0$ and this contradiction (u_i/U_i^0 is T-periodic) finishes the proof.

PROPOSITION. *Let u, t_0, $U_1, ..., U_n$ be as in Proposition* 2.1 *and suppose that there is an $\varepsilon>0$ such that*

$$b_i(t)>\varepsilon a_{ii}(t)+\sum_{j\in J_i} a_{ij}(t)\,U_j(t); \qquad 1\leqslant i\leqslant n,\ t\geqslant y_0 \tag{2.1}$$

then $u_i(t)>\min\{u_i(t_0), \varepsilon\}$; $t>t_0$; $1\leqslant i\leqslant n$.

Proof. Let us define $\varepsilon_i=\min\{u_i(t_0), \varepsilon\}$ and $v_i(t)=\varepsilon_i-u_i(t)$ $(t\geqslant t_0)$. We shall prove that $v_i\leqslant 0$ in $[t_0, \infty)$. To prove this let us assume that $v_i(t_2)>0$ for some $t_2>t_0$; since $v_i(t_0)\leqslant 0$, there is $t_1>t_0$ such that $v_i(t_1)>0$ and $v_i'(t_1)>0$; from this and Proposition 2.1 we get

$$b_i(t_1)<\sum_{j=1}^{n} a_{ij}(t_2)\,u_j(t_1)<\varepsilon_i a_{ii}(t_1)+\sum_{j\in J_i} a_{ij}(t_1)\,U_j(t_1)$$

which contradicts (2.1) and proves that $u_i(t) \geqslant \varepsilon_i$ if $t \geqslant t_0$; $1 \leqslant i \leqslant n$. Assume now that $u_i(t_1) = \varepsilon_i$ for some $t_1 > t_0$; then $u_i'(t_1) = 0$ and as above we get a contradiction. So the proof is finished.

From now on, $\mathbb{R}^n_+$ denotes the set of points x in $\mathbb{R}^n$ such that $x > 0$. Moreover $u(t, x) = (u_1(t, x), ..., u_n(t, x))$ denotes the solution of (0.1) defined by the initial condition $u(0, x) = x$. Remember that $u(t, x)$ is defined in $[0, \infty)$.

2.3. THEOREM. *If the conditions* (0.5) *are satisfied then the system* (0.1) *has at least a positive and T-periodic solution* u^0 *such that* $u^0(t) > 0$ *for all* $t \in \mathbb{R}$. *Moreover, for each compact set* $K \subseteq \mathbb{R}^n_+$ *there are* ε_k, $M_k > 0$ *such that* $\varepsilon_k \leqslant u_i(t, x) \leqslant M_k$ *if* $t \geqslant 0$ *and* $x \in K$.

Proof. Since (0.5) is satisfied then there is $\varepsilon > 0$; $\varepsilon \leqslant \min\{U_i^0(0)$: $1 \leqslant i \leqslant n\}$, such that

$$b_i(t) > \varepsilon a_{ii}(t) + \sum_{j \in J_i} a_{ij}(t)[\varepsilon + u_j^0(t)]; \qquad t \in \mathbb{R}; \quad 1 \leqslant i \leqslant n. \tag{2.2}$$

Let us define $A = \{x = (x_1, ..., x_n) \in \mathbb{R}^n$: $\varepsilon \leqslant x_i \leqslant U_i^0(0)$; $1 \leqslant i \leqslant n\}$; from Propositions 2.1 and 2.2 it follows that the map $F: A \to A$; $F(x) = u(T, x)$; is well defined. Remember that U_i^0 is T-periodic; $1 \leqslant i \leqslant n$. Since F is continuous then there exists $x_0 \in A$ such that $F(x_0) = x_0$; so $u^0(t) := u(t, x_0)$ is a T-periodic solution of (0.1) such that $u^0(t) > 0$ for each $t \in \mathbb{R}$.

Let $K \subset \mathbb{R}^n_+$ be compact set and for each $x = (x_1, ..., x_n)$ let us denote by $U_i(t, x)$ the solution of $(0.4)_i$ given by $U_i(0, x) = x_i$. It is not hard to prove that $U_i(t, x) - U_i^0(t) \to 0$ (as $t \to +\infty$) uniformly for $x \in K$ $(1 \leqslant i \leqslant n)$; consequently, there is $t_0 \geqslant 0$ such that $U_i(t, x) \leqslant U_i^0(t) + \varepsilon$ if $t \geqslant t_0$, $x \in K$, and $1 \leqslant i \leqslant n$. Thus, by Proposition 2.1, one has $u_i(t, x) \geqslant \min\{u_i(t_0, x), \varepsilon\}$ $(t \geqslant t_0,\ 1 \leqslant i \leqslant n,\ x \in K)$ and hence there is an $\varepsilon_k > 0$ such that $u_i(t, x) \geqslant \varepsilon_k$ $(t \geqslant 0,\ 1 \leqslant i \leqslant n,\ x \in K)$. On the other hand $u_i(t, x) \leqslant U_i(t, x) \leqslant \varepsilon + U_i^0(t)$ $(t \geqslant t_0,\ x \in K,\ 1 \leqslant i \leqslant n)$ and the proof follows easily.

3. UNIQUENESS AND ASYMPTOTICITY

In this section we prove basically the second part of Theorem 0.1. To prove this we shall need the following version of a theorem of Lazer [5].

3.1. THEOREM. *Let* $B(t) = (b_{ij}(t))$ *be a real functional* $n \times n$ *matrix defined and continuous in* $[0, \infty)$, *such that* $b_{ij}(t) > 0$; $1 \leqslant i, j \leqslant n$, $t \geqslant 0$. *Suppose further that there is* $\delta > 0$ *such that*

$$b_{ii}(t) > \delta + \sum_{j \in J_i} b_{ij}(t); \qquad 1 \leqslant i \leqslant n; \quad t \geqslant 0. \tag{3.1}$$

If $\Gamma(t)$ *denotes the fundamental matrix of the system* $x' = B(t)x$; $\Gamma(0) = $ *identity*; *then*

$$\|\Gamma(t)w\|_\infty \geqslant e^{\delta t}\|w\|_\infty; \qquad t \geqslant 0, \quad w \in \mathbb{R}^n,$$

where $\|w\|_\infty = \max\{|w_1|, ..., |w_n|\}$ *if* $w = (w_1, ..., w_n)$.

In the next result we do not assume periodicity conditions on the coefficients bi and a_{ij} $(1 \leqslant i, j \leqslant n)$.

3.2. THEOREM. *Suppose that there are positive constants* $\alpha_1, ..., \alpha_n$, δ_1 *such that*

$$\alpha_i a_{ij}(t) > \delta_1 + \sum_{j \in J_i} \alpha_j a_{ji}(t); \qquad t \geqslant 0; \quad 1 \leqslant i \leqslant n. \tag{3.2}$$

If $K \subset \mathbb{R}^n_+$ *is a convex set and there are positive constants* ε_k, M_k *such that* $\varepsilon_k \leqslant u_i(t, x) \leqslant M_k$ *for* $t \geqslant 0$, $x \in K$, $1 \leqslant i \leqslant n$, *then there are positive constants* δ, k *such that*

$$\|u(t, x) - u(t, y)\| \leqslant ke^{-\delta t}\|x - y\| \qquad \text{if} \quad t \geqslant 0; \quad x, y \in K.$$

where $\|\cdot\|$ is the usual Euclidean norm of $\mathbb{R}^n$.

Proof. Let us fix $x \in K$; we know that the partial derivative $u_x(t, x)$ is the fundamental matrix Φ_x; $\Phi_x(0) = $ identity; of the linear system

$$y_i' = \frac{u_i'(t, x)}{u_i(t, x)} y_i - \sum_{j=1}^n a_{ij}(t)\, u_i(t, x)\, y_j; \qquad 1 \leqslant i \leqslant n.$$

From the change of variable $z_i = y_i/u_i(t, x)$ we get

$$z_i' = -\sum_{j=1}^n a_{ij}(t)\, u_j(t, x)\, z_j. \tag{3.3}$$

Notice that, if ψ_x is the fundamental matrix of (3.3) with $\psi_x(0) = $ identity, then $\Phi_x(t) = D_x(t)\, \psi_x(t)\, D_x(0)^{-1}$ where $D_x(t)$ is the diagonal matrix, $\operatorname{diag}(u_1(t, x), ..., u_n(t, x))$.

Let $\Lambda_x(t)$ be the fundamental matrix, with $\Lambda_x(0) = $ identity, of the adjoint system to (3.3),

$$z_i' = \sum_{j=1}^n a_{ji}(t)\, u_i(t, x)\, z_j; \qquad 1 \leqslant i \leqslant n \tag{3.4}$$

and let D_α be the diagonal matrix $\operatorname{diag}(\alpha_1, ..., \alpha_n)$. Then $\Gamma_x(t) = D_\alpha^{-1} \circ \Lambda_x(t) \circ D_\alpha$ is the fundamental matrix ($\Gamma_x(0) = $ identity) of the linear system $x' = B(t)x$, where $B(t) = (b_{ij}(t))$ and $b_{ij}(t) = \alpha_j \alpha_i^{-1} u_i(t, x)\, a_{ji}(t)$.

On the other hand

$$b_{ii}(t) > \delta + \sum_{j \in J_i} b_{ij}(t); \qquad t \geqslant 0, \quad 1 \leqslant i \leqslant n,$$

where $\delta = \varepsilon_k \, \delta_1 \min\{\alpha_1^{-1}, ..., \alpha_n^{-1}\}$. By Theorem 3.1 we have $\|\Gamma_x(t) w\|_\infty \geqslant e^{\delta t} \|w\|_\infty$ $(t \geqslant 0,\ x \in K,\ w \in \mathbb{R}^n)$. From this, it is easy to prove that there is a positive constant such that $\|A_x(t)^{-1} w\| \leqslant k_1 e^{-\delta t} \|w\|$ $(t \geqslant 0, x \in K, w \in \mathbb{R}^n)$ and hence $\|\psi_x(t) w\| \leqslant k_1 e^{-\delta t} \|w\|$ $(t \geqslant 0,\ x \in K,\ w \in \mathbb{R}^n)$. Therefore, there is $k > 0$ such that $\|\Phi_x(t) w\| \leqslant k e^{-\delta t} \|w\|$ $(t \geqslant 0,\ x \in K,\ w \in \mathbb{R}^n)$ and the proof follows from the integral value theorem:

$$u(t, x) - u(t, y) = \int_0^1 u_x(t, (1-s) y + sx)(x - y) \, ds.$$

Proof of Theorem 0.1. In Theorem 2.3 we have proved that the system (0.1) has at least a T-periodic and positive solution u^0. On the other hand, conditions (3.2) are satisfied for some $\delta_1 > 0$ because a_{ij} is T-periodic for all $1 \leqslant i, j \leqslant n$. Now let $K \subset \mathbb{R}_+^n$ be a compact set. Without loss of generality, we can assume that K is a convex set (it suffices to consider the convex hull of K). And from Theorem 2.3 we get two positive constants ε_k, M_k such that $\varepsilon_k \leqslant u_i(t, x) \leqslant M_k$ if $x \in K$; $1 \leqslant i \leqslant n$ and $t \geqslant 0$. The proof follows now from Theorem 3.2.

References

1. S. Admad, On almost periodic solutions of the competing species problems, to appear.
2. C. Alvarez and A. Lazer, An application of topological degree to the periodic competing species problem, *J. Austral. Math. Soc. Ser. B* **28** (1986), 202–219.
3. R. Bellman, "Introduction to Matrix Analysis," McGraw-Hill, New York, 1960.
4. K. Gopalsamy, Global asymptotic stability in a periodic Lotka–Volterra system, *J. Austral. Math. Soc. Ser. B* **27** (1985), 66–72.
5. A. Lazer, Characteristic exponents and diagonal dominant linear systems, *J. Math. Anal. Appl.* **35** (1971), 215–229.